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ON THE CONDITIONS FOR THE ONSET OF MOTION OF TWO COLLINEAR DISLOCATION DISCONTINUITIES*

A.S. BYKOVTSEV

The conditions under which the motion begins of two collinear dislocational Volterra-type discontinuities, initally specified on a single straight line in a homogeneously isotropic elastic medium, is studied. The theory of invariant Γ-integrals /1/ is used to write the criteria defining the beginning and direction of motion of either end of the discontinuity. The limiting stresses are determined and the subsequent behaviour of the whole system is investigated.

Let two generalized dislocational discontinuities of unequal length and constant sudden change in displacement $\mathbf{b}(b_1, b_2, b_3) = \text{const}$ be distributed along a single straight line. We introduce the rectangular Cartesian coordinate system in such a manner that the Ox-axis coincides with the line on which the discontinuities lie, and denote by $-l_1, -l_2, l_3, l_4$ the abscissas of the ends of the discontinuity. The problem is assumed to be plane. We will determine the critical loads which must be applied to the body in order for at least one end of the discontinuity to begin to move. The problem in question is an analog of the problem discussed in /2/ (on the equilibrium of two collinear cracks) for dislocation discontinuities.

Let us denote by u_x, u_y, u_z the components of the displacement vector along the x, y, z axes respectively, and by $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}$ the stress tensor components. We also denote the set of internal points of the segments $(-l_1, -l_2)$ and (l_3, l_4) of the Ox-axis by L, and the set of points of the Ox-axis outside these segments by M. The boundary conditions of the problem have the form

$$[\mathbf{u}] = \mathbf{b} \text{ on } L, \ [\mathbf{u}] = 0 \text{ on } M \tag{1}$$

Problem (1) can be written in the form of the sum of the symmetric, skew-symmetric and anti-plane problems, by expanding the vector $\mathbf{b}(b_1, b_2, b_3)$ in three terms $\mathbf{b}_1(b_1, 0, 0), \mathbf{b}_2(0, b_2, 0), \mathbf{b}_3(0, 0, b_3)$. The boundary conditions will have the form (2), (3) and (4) for the skew-symmetric, symmetric and antiplane problems respectively

$$u_x = {}^1/_2 b_1, \ \sigma_{yy} = 0 \text{ on } L; \ u_x = 0, \ \sigma_{yy} = 0 \text{ on } M \tag{2}$$

$$u_y = {}^1/_2 b_2, \ \sigma_{xy} = 0 \text{ on } L; \ u_y = 0, \ \sigma_{xy} = 0 \text{ on } H \tag{3}$$

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$$u_z = {}^1/_2\, b_3 \text{ on } L;\ u_z = 0 \text{ on } M \tag{4}$$

We will write the general representation of the solutions of the equilibrium equations in terms of the complex potentials $W_i(z)$ $(i = 1, 2, 3)$ [3]: thus:

$$\begin{aligned} 2Gu_x &= {}^1/_2 (\varkappa + 1) \operatorname{Im} w_1 + y \operatorname{Re} W_1 \\ 2Gu_y &= -(\varkappa + 1) \operatorname{Re} w_1 - y \operatorname{Im} W_1 \\ \sigma_{xx} &= 2\operatorname{Im} W_1 + y \operatorname{Re} W_1', \quad \sigma_{yy} = -y \operatorname{Re} W_1' \\ \sigma_{xy} &= \operatorname{Re} W_1 - y \operatorname{Im} W' \end{aligned} \tag{5}$$

for skew-symmetric problems,

$$\begin{aligned} 2Gu_x &= {}^1/_2 (\varkappa - 1) \operatorname{Re} w_2 - y \operatorname{Im} W_2 \\ 2Gu_y &= {}^1/_2 (\varkappa + 1) \operatorname{Im} w_2 - y \operatorname{Re} W_2 \\ \sigma_{xx} &= \operatorname{Re} W_2 - y \operatorname{Im} W_2', \quad \sigma_{xy} = -y \operatorname{Re} W_2' \\ \sigma_{yy} &= \operatorname{Re} W_2 + y \operatorname{Im} W_2' \end{aligned} \tag{6}$$

for symmetric problems and

$$u_z = \operatorname{Re} w_3, \quad \sigma_{xz} = G \operatorname{Re} W_3, \quad \sigma_{yz} = -G \operatorname{Im} W_3 \tag{7}$$

for the antiplane case. Here $w_i'(z) = W_i(z)$ $(i = 1, 2, 3)$; $\varkappa = 3-4\nu$ for a plane deformation, $\varkappa = (3-4\nu)/(1+\nu)$ for the generalized plane state of stess, G is the shear modulus, ν is Poisson's ratio, and the prime denotes differentiation with respect to z.

From (2)—(7) we obtain the Dirichlet problem

$$\begin{gathered} \operatorname{Im} w_i = A_i, \quad z \in L; \ \operatorname{Im} w_i = 0, \quad z \in M \\ A_i = \frac{2Gb_i}{\pi(\varkappa+1)} \quad (i = 1,2), \quad A_3 = \frac{Gb_3}{2\pi} \end{gathered} \tag{8}$$

The solution of (8) has the form

$$w_i = A_i \ln \frac{(l_2 + z)(l_4 - z)}{(l_1 + z)(l_3 - z)} \tag{9}$$

Substituting (9) into (5)—(7) we obtain the following stress field for the generalized dislocation discontinuities:

$$\sigma = \sigma(x - l_4, y) - \sigma(x - l_3, y) + \sigma(x + l_2, y) - \sigma(x + l_1, y) \tag{10}$$

where the components of the stresses $\sigma(x, y)$ have the form

$$\begin{gathered} \sigma_{xx} = r^{-1} [-A_1 \sin\varphi\, (2 + \cos 2\varphi) + A_2 \cos\varphi \cos 2\varphi] \\ \sigma_{yy} = r^{-1} [-A_1 \sin\varphi \cos 2\varphi + A_2 \cos\varphi\, (2 - \cos 2\varphi)] \\ \sigma_{xy} = r^{-1} \cos 2\varphi\, (A_1 \cos\varphi + A_2 \sin\varphi), \quad \sigma_{zy} = r^{-1} A_3 \cos\varphi \\ \sigma_{zx} = -r^{-1} A_3 \sin\varphi \\ r = \sqrt{x^2 + y^2}, \quad \varphi = \operatorname{arctg} \frac{y}{x} \end{gathered} \tag{11}$$

The onset and direction of the motion of any end of the dislocation discontinuities will be found using the invariant Γ- integrals /1/

$$\Gamma_k = \int_\Sigma [Un_k - \sigma_{ij} u_{i,k} n_j]\, d\Sigma \tag{12}$$

Here U is the elastic potential of unit volume, Σ is an arbitrary small contour embracing the end of the dislocational discontinuity in question, n_k are the components of the unit vector normal to the contour, and double indexing with the obvious meaning $(1, 2, 3) \to (x, y, z)$ is used.

We assume that an external stress field with components σ_{ij}° is applied to the body at infinity. Then relations (10)—(12) yield the following expressions for the Γ-integrals for every end of the dislocational discontinuities:

$$\begin{aligned} \Gamma_x^{\,l_i} &= b_1 (\sigma_{xy}^\circ + A_1 \Gamma^{l_i}) + b_2 (\sigma_{yy}^\circ + A_2 \Gamma^{l_i}) + b_3 (\sigma_{yz}^\circ + A_3 \Gamma^{l_i}) \\ \Gamma_y^{\,l_i} &= -b_1 (\sigma_{xx}^\circ + A_2 \Gamma^{l_i}) - b_2 (\sigma_{xy}^\circ + A_1 \Gamma^{l_i}) - b_3 \sigma_{zx}^\circ \end{aligned} \tag{13}$$

where

$$\begin{gathered} \Gamma^{-l_1} = -\frac{1}{a_1} + \frac{a_2}{(a_1 + a_2 + a_3)(a_1 + a_3)}, \quad \Gamma^{-l_2} = -\frac{1}{a_1} + \frac{a_2}{a_3 (a_2 + a_3)} \\ \Gamma^{l_4} = -\frac{1}{a_2} + \frac{a_1}{(a_1 + a_2 + a_3)(a_2 + a_3)}, \quad \Gamma^{l_3} = -\frac{1}{a_2} + \frac{a_1}{a_3 (a_1 + a_3)} \\ a_1 = -l_2 + l_1, \quad a_2 = l_4 - l_3, \quad a_3 = l_3 + l_2 \end{gathered} \tag{14}$$

Here a_1 and a_2 denote the lengths of the left and right discontinuity while a_3 is the distance between them.

The condition for the onset of motion of the end of a discontinuity can be written in the form $|\Gamma| = \Gamma_c$ (when $|\Gamma| < \Gamma_c$, the end of the discontinuity will be at rest). Here $\Gamma = \Gamma_x \mathbf{i} + \Gamma_y \mathbf{j}$, Γ_c is the experimentally determined constant of the medium.

The direction of motion of the ends of the discontinuity is given by the expression

$$\theta_{l_i} = \operatorname{arctg}\,(\Gamma_y{}^{l_i}/\Gamma_x{}^{l_i})$$

Let us study in more detail the case when $b_1 \neq 0$, $b_2 = b_3 = 0$, $\sigma_{xy}{}^\circ \neq 0$, $\sigma_{yy}{}^\circ = \sigma_{yz}{}^\circ = \sigma_{xx}{}^\circ = \sigma_{zz}{}^\circ = 0$, i.e., the skew-symmetric discontinuities are acted upon by tangential stresses only. Such discontinuities are of interest in theoretical seismology in modelling the processes taking place in the danger zone prior to an earthquake. In this case we find that

$$\Gamma_y{}^{l_i} = 0,$$

for all tips, i.e. at the initial stage the discontinuities can move only along the Ox-axis on which they are situated.

Relations (13) and (14) show that the values of the Γ-integrals depend essentially on the size of the discontinuities and the distance between them. Let us assume that $a_1 \gg a_2$, i.e. consider the case when a finite discontinuity interacts with a semi-infinite one. Then from (14) we obtain

$$\Gamma^{-l_1} \approx 0, \quad \Gamma^{-l_2} = \frac{a_2}{a_3(a_2+a_3)} > 0, \quad \Gamma^{l_4} = -\frac{a_2}{a_2(a_2+a_3)} < 0$$
$$\Gamma^{l_3} = \frac{a_2 - a_3}{a_2 a_3} \begin{cases} > 0, & a_2 > a_3 \\ \leqslant 0, & a_2 \leqslant a_3 \end{cases} \tag{15}$$

and from (13)

$$\Gamma_x^{l_i} = b_1(\sigma_{xy}{}^\circ + A_1 \Gamma^{l_i})$$

The estimates (15) show clearly that when $\sigma_{xy}{}^\circ = 0$, the criterion for the onset of propagation of the discontinuity (depending on the parameters of the discontinuities themselves)

$$\Gamma_c = b_1 A_1 \Gamma^{l_i} \tag{16}$$

will hold, in the first instance, for the end with abscissa $-l_2$, then for the point l_3, and finally for l_4. If on the other hand the discontinuity parameters are such that the criterion (16) does not hold when $\sigma_{xy}{}^\circ = 0$, then the body may be in a state of equilibrium provided that additional stresses $\sigma_{xy}{}^\circ$ are applied to it. Then the magnitude of the limiting stresses $\sigma_{xy}{}^*$, for which it becomes possible that the ends with abscissa $-l_2$ most predisposed to move will do so, is given by the relation

$$\sigma_{xy}{}^* = (\Gamma_c - b_1 A_1 \Gamma^{-l_2})/b_1 \tag{17}$$

Thus the interaction between a semi-infinite discontinuity with a discontinuity of finite length can be described as follows. When the external stress field $\sigma_{xy}{}^\circ$ is increased, the first to move will be the end of the semi-infinite discontinuty (when the stresses reach the value $\sigma_{xy}{}^*$ given by the formula (17)). This end will move towards the finite discontinuity, the distance between the ends of the discontinuities will decrease and hence Γ^{l_3} will increase. It follows that at some instant when the condition

$$\sigma_{xy} = (\Gamma_c - b_1 A_1 \Gamma^{l_3})/b_1$$

begins to hold, the left end of the small discontinuity will begin to move towards the moving end of the semi-infinite discontinuity. It is only after both ends merge (i.e. after the barier separating them is breached), that motion of the right end of the smaller discontinuity will become possible.

The rate of motion of one end relative to the other will increase in a step-wise manner. This implies that, in particular, the rate of joining or merging of the discontinuities may be greater than the velocity of the longitudinal and transverse waves within the medium. Finally, we find that when we have a large and a small discontinuity separated by an arbitrarily large distance, the larger discontinuity will always show a tendency to merge with the smaller one. At the same time, the small discontinuity will behave with complete independence until the distance separating the discontinuities becomes less than its length, whereupon it will begin to move towards a merger with the large discontinuity. The effect of the interaction between the dislocation discontinuities is analogous to that of the interaction between collinear cracks /2/ and serves as another obvious confirmation of the fact that the behaviour of the dislocational discontinuities is formally analogous to the behaviour of cracks, a fact demonstrated by the analogs of the Griffith and Yoffe problems for dislocational discontinuities in /4—6/.

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ON THE REMAINDER TERMS IN THE FORMULAS FOR THE FREQUENCY DISTRIBUTION OF SHELL OSCILLATIONS*

A.G. ASLANIAN

The frequency distribution of free oscillations of thin elastic shells in vacuo and in contact with a liquid, is studied. Estimates for the remainder terms in the asymptotic formulas for the oscillation frequency distribution are substantially improved. In the case of a shell in contact with a liquid, the second term of the asymptotics is separated.

Free oscillations of a thin elastic shell are described by a system of three differential equations in terms of the displacements /1/

$$(1/12h^2N + L)\,u = \lambda u,\ \lambda = (1-\sigma^2)\,\rho E^{-1}\omega^2 \tag{1}$$

The vector function $u(\alpha,\beta)$, $(\alpha,\beta)\in G$ satisfies certain selfconjugate boundary conditions at the shell boundary, h is the shell thickness (small parameter), λ is the spectral parameter and ω is the natural shell oscillation frequency. The remaining notation is taken from /1/.

Let $n_h(\lambda)$ be the spectrum distribution function of problem (1) (equal to the number of eigenvalues less than the given λ). Using the variational method as $h\to+0$ we obtain /1/ the asymptotic formula

$$n_h(\lambda) = h^{-1}[c_0(\lambda) + O(h^{\varkappa})] \tag{2}$$

$$c_0(\lambda) = \frac{\sqrt{3}}{4\pi^2}\iint_G\int_0^{2\pi}\operatorname{Re}(\lambda - \Omega(\theta,\alpha,\beta))^{1/2}\,d\theta\,dS$$

$$\Omega(\theta,\alpha,\beta) = (1-\sigma^2)\,[R_1^{-1}(\alpha,\beta)\sin^2\theta + R_2^{-1}(\alpha,\beta)\cos^2\theta]^2$$

where $\varkappa$ is a positive number. A rough lower estimate was given for it in /1,2/.

By improving the variational technique, we succeeded in showing that when

$$\lambda > \sup\Omega(\theta,\alpha,\beta),\ \theta\in[0,2\pi],\ (\alpha,\beta)\in G \tag{3}$$

formula (2) holds with $\varkappa = 1/_2 - \varepsilon$, for arbitrarily small positive ε.

If condition (3) does not hold, then the value of $\varkappa$ decreases and depends on the amount of "degeneration" of the function $q = \lambda - \Omega(\theta,\alpha,\beta)$. For example, if $q^{-1/2}$ is integrable, $\theta\in[0,2\pi]$, $(\alpha,\beta)\in G$ (simple degeneration), formula (2) holds for $\varkappa = 5/22$.

In the same manner we can improve the estimate of the remainder in the problem of free oscillations of a shell in contact with an ideal compressible liquid /3/. In this problem we add to the right-hand side of the third equation of (1) the term $-h^{-1}\rho_f\rho_0^{-1}\lambda\varphi|_S$ (the inertia of the liquid). The potential $\varphi(x,y,z)$ of the displacement of the liquid occupying a finite volume V, satisfies the Helmholtz equation

$$\Delta\varphi + k_0\lambda\varphi = 0,\ (x,y,z)\in V,\ k_0 = \frac{E}{(1-\sigma^2)\rho_0 c_f^2}$$

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